A FEW PROBLEMS ON MONODROMY AND DISCRIMINANTS

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1. Explicit obstructions to the Lyashko–Looijenga covering (and its real analogs) for non-simple singularities

The so-called Lyashko–Looijenga covering (see [6], [15]) is a strong tool for constructing (or proving the existence of) the perturbations of simple singularities with prescribed topological properties, such as singularity types of different critical points, or intersection matrices of vanishing cycles, see e.g. [8]. The real version of this tool allows one to construct and enumerate all topologically different Morsifications of real simple singularities, see [7], [4], [18].

A large amount of these options is preserved for non-simple singularities, see [18], [15]. In particular, this method has predicted the existence of many Morsifications with prescribed properties and indicated their topological characteristics, so that it was easy to give a strict construction of these Morsifications. However, in this case this method is rater experimental or heuristic, without clear guaranties that all perturbations found by it actually do exist. Therefore it is important to fix the restrictions of this method. Here are several explicit problems.

1.1. **Complex version.** Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^1, 0)$ be an isolated holomorphic function singularity, μ its Milnor number, $F(x, \lambda) : (\mathbb{C}^n \times \mathbb{C}^\mu, 0) \to (\mathbb{C}^1, 0)$ the miniversal deformation of f, and $\Sigma \subset \mathbb{C}^\mu$ the complete bifurcation set of functions of this deformation, i.e. the set of values of the parameter $\lambda \in \mathbb{C}^\mu$ such that the corresponding function $f_\lambda \equiv F(\cdot, \lambda)$ has less than μ different critical values at critical points close to the origin. The *Lyashko–Looijenga map* sends any point λ from a small neighborhood B_ε of the origin in $\mathbb{C}^\mu$ to the unordered collection of critical values of the function f_λ at points close to $0 \in \mathbb{C}^n$ (or, which is equivalent but sometimes more convenient, to the set of values of basic symmetric polynomials of these critical values). If the singularity

of f is simple then the restriction of this map to $B_\varepsilon \setminus \Sigma$ defines a local covering over the configuration space $B(D, \mu)$ of all subsets of cardinality μ in a very small (even with respect to ε) neighborhood D of the origin in $\mathbb{C}$, see [6]. In particular, any element $\alpha \in \pi_1(B(D, \mu))$ can be realised by a loop which can be lifted to a path in $B_\varepsilon \setminus \Sigma$ covering this loop.

For non-simple singularities this is not more the case. As previously, the Lyashko–Looijenga map is submersive (and hence locally bijective) everywhere in $B_\varepsilon \setminus \Sigma$ (this follows from the very notion of versality). However, a sufficiently complicated path in $B(D, \mu)$, being lifted into $\mathbb{C}^\mu \setminus \Sigma$ in accordance with this local bijectivity, can run out from the neighborhood of the origin in $\mathbb{C}^\mu$. This is related with the fact that for non-simple singularities the Lyashko–Looijenga map is not proper: the preimage of the collection $(0, \ldots, 0)$ is the entire (positive-dimensional) $\mu = const$ stratum[1].

Problem 1A: *to present explicit obstructions to the Lyashko-Looijenga covering in the terms of braid groups. Which braids cannot be lifted to the space $\mathbb{C}^\mu \setminus \Sigma$?*

Given a configuration of μ different points $z_1, \ldots, z_\mu$ in $D \setminus 0$ and a system of non-intersecting paths connecting them to 0, any perturbation f_λ of f, having these critical values, defines a *Dynkin diagram,* see [1], vol. 2. Any braid $l \in \pi_1(B(D, \mu))$ moves this Dynkin diagram to another one in accordance with the Picard–Lefschetz formulas (see [1] or [15]). If our braid l can be lifted to a curve in $B_\varepsilon \setminus \Sigma$ starting at the point λ and covering this braid via the Lyashko–Looijenga map, then the resulting Dynkin diagram is nothing else than the Dynkin diagram of the function $f_{\lambda'}$ corresponding to the endpoint of this lifted curve and defined by the same system of paths connecting the critical values to 0.

For complicated singularities the number of Dynkin graphs which can be achieved by the formal Picard-Lefschetz moves is infinite, while the number of preimages of any non-discriminant configuration under the Lyashko–Looijenga map is bounded.

Problem 1B: *given a non-simple singularity and a Dynkin diagram of it defined by an easy distinguished system of paths connecting 0 to critical points of f_λ, which Dynkin graphs can be achieved from it by*

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[1]A weaker substitute for the Lyashko–Looijenga covering theorem holds in the case of parabolic singularities, if one writes the versal deformation in the canonical monomial form and allows large travellings in the space $\mathbb{C}^\mu$, see [5]. For more complicated singularities the situation is even worse.

a sequence of formal Picard-Lefschetz moves defined by a braid, but cannot appear as Dynkin diagrams of Morsifications $f_{\lambda'}$ with the same critical values, defined by the same system of paths?

Further, for simple singularities all partial collisions of μ critical values can be realized, because the Lyashko–Looijenga map is proper. This reduces the problem of the enumeration of possible decompositions of the initial critical point to a problem formulated in the terms of Dynkin diagrams and Picard–Lefschetz operators only, see [8]. Again, for non-simple singularities it is not the case. For instance, any non-simple singularity admits a system of paths, connecting 0 to critical values, such that the intersection index of some two vanishing cycles is equal to ± 2. Then we surely cannot lift to B_ε the collision of these two critical values along these paths (keeping the remaining critical values unmoved). Namely, the attempt to move these critical values towards one another by means of the Lyashko–Looijenga submersion will throw the parameter λ from any neighborhood of the origin in $\mathbb{C}^\mu$.

Problem 1C. *Are there more refined restrictions to the collision of critical values? Is it correct that if the intersection index of some two vanishing cycles is equal to ± 1 or 0, then we can lift the collision of the corresponding critical values to B_ε via the Lyashko–Looijenga submersion?*

In the previous consideration, the existence of two vanishing cycles with intersection index ± 2 ensures the non-properness of the Lyashko–Looijenga map, and hence the fact that the $\mu = const$ stratum of the singularity is positive-dimensional.

Problem 1D. *Give more general lower bounds of the dimension of $\mu = const$ strata in the terms of intersection forms of vanishing cycles.*

That is, if we can indicate many independent prohibited collisions of critical values, then probably the attempt to perform these collisions by the rough force will throw us from the neighborhood of the origin in $\mathbb{C}^\mu$ in independent directions (all of which approach the $\mu = const$ stratum).

1.2. **Real version.** The real versions of these problems are important for the construction of real decompositions and enumeration of topologically distinct Morsifications of real singularities, see [4], [18]. Namely, let $f : (\mathbb{C}^n, \mathbb{R}^n, 0) \to (\mathbb{C}, \mathbb{R}, 0)$ be a real function singularity, and $F : (\mathbb{C}^n \times \mathbb{C}^k) \to \mathbb{C}$ its real deformation (that is, $F(x, \lambda)$ is real if $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^k \subset \mathbb{C}^k$). The space $\mathbb{R}^k$ of real parameters is separated into several chambers by the real total discriminant (consisting of all non-Morse functions and functions with critical value 0). We

can go from any chamber to any other one by a generic path in $\mathbb{R}^k$, passing only finitely many times the discriminant at its non-singular point. Any such passage changes the topological type of the function f_λ in some predictable way. Moreover, if our singularity f is simple and the deformation F is versal, then all standard changes satisfying some natural restrictions can be indeed performed. In particular, if f_λ has two neighboring real critical values, then we can collide them and get two critical points on the same level (if the intersection index of corresponding vanishing cycles is equal to 0) or a critical point of type A_2 (if this index is equal to ± 1); in the latter case these two critical values (and the corresponding critical points) go to the imaginary domain after this passage.

For non-simple singularities, we can perform all the same formal surgeries over the collections of critical values (supplied with the intersection matrix and some additional set of topological invariants of a real Morsification), and combine these formal surgeries in arbitrary sequences.

Problem 1E. *What are the obstructions to the realization of these chains of formal changes by paths in the parameter space $\mathbb{R}^k$?*

An algorithm enumerating all such chains of surgeries was realized in [18]; at least for singularities of corank 2 and $\mu \leq 11$ it never met a formal surgery which could not be realized by a surgery of functions in the versal deformation.

Can this experimental fact be raised to the theorem level?

Here is a particular problem, needed for some improvement of this algorithm in the case of singularities of corank ≥ 3.

1.3. **Prediction of the indices of newborn critical points at a Morse surgery.** Consider an one-parametric family of real analytic functions (or just polynomials) $f_\tau : (\mathbb{C}^n, \mathbb{R}^n) \to (\mathbb{C}, \mathbb{R})$, $\tau \in (-\varepsilon, \varepsilon)$ realizing a Morse birth surgery: the functions f_τ, $\tau < 0$, have two complex conjugate critical points which collide in a point of type A_2 when τ tends to 0, and after that reappear as two real Morse critical points of some two neighboring Morse indices.

Problem 1F: *is there any convenient topological characteristic of the function $f_{-\varepsilon}$ which allows us to predict these indices?*

The *parities* of these indices can be indeed predicted. Namely, consider the complex level manifold $V_a \equiv f_{-\varepsilon}^{-1}(a)$, where a is a real non-critical value between the (complex conjugate) critical values of $f_{-\varepsilon}$ which are going to collide, and vanishing cycles in this manifold defined

by segments connecting a with these critical values. The intersection index of these cycles is equal to ± 1 depending on the choice of their orientations. Let us choose these orientations in such a way that the complex conjugation in V_a takes one of them into the other. Then the sign of their intersection number is well-defined and allows us to guess the parity of the greater newborn critical point, see [15], [18]. But how can we predict the integer index?

2. COVERING NUMBER (GENUS) OF MAPS WHICH ARE NOT FIBER BUNDLES

Given a surjective map of topological spaces, $p : X \to Y$, the *covering number* of p is the minimal number of open sets covering Y in such a way that there is a cross-section of p over any of these sets. This definition was given by S. Smale [14] in connection with the problems of complexity theory. In the particular case of fiber bundles, this notion was earlier introduced and deeply studied by A.S. Schwarz [11] under the name of the *genus* of a fiber bundle. However, in the complexity theory of equations over real numbers, the case of maps with varying fibers becomes essential. Here is one of the first examples. Consider the 6-dimensional real space of pairs of polynomials $(f_a, g_b) : \mathbb{R}^2 \to \mathbb{R}^2$, where $f_a(x, y) = x^2 - y^2 + a(x, y)$, $g_b = xy + b(x, y)$, $a(x, y)$ and $b(x, y)$ are arbitrary polynomials of degree ≤ 1. Obviously, the system $\{f_a = 0, g_b = 0\}$ always has 2 or 4 solutions in $\mathbb{R}^2$ (counted with multiplicities).

Problem 2A. *What is the minimal number of open sets U_i covering $\mathbb{R}^6$ such that for any U_i there is a continuous map $\varphi_i : U_i \to \mathbb{R}^2$ sending any pair $(a, b) \in U_i$ into some solution of the system (f_a, g_b)?*

In the previous terms, this is the question about the covering number of the projection map $X \to Y$, where $Y = \mathbb{R}^6$ is the space of parameters (a, b), and $X \subset \mathbb{R}^6 \times \mathbb{R}^2$ is the space of pairs $((a, b), (x, y))$ such that $(x, y) \in \mathbb{R}^2$ is a root of the system (f_a, g_b).

The number in question is not less than 2 (indeed, we can emulate the complex equation $z^2 = A$ inside our system, and the covering number of this equation depending on the complex parameter A is equal to 2). But is this estimate sharp?

Problem 2B. *The same questions concerning the approximate solutions. That is, for any i and $(a, b) \in U_i$ the value $\varphi_i(a, b)$ should be not necessarily a root of (f_a, g_b), but just a point in the ε–neighborhood of such a root for some fixed positive ε.*

These problems have obvious generalizations to polynomial systems of higher degrees and different numbers of variables. They can be non-trivial already in the case of polynomials (1) in one real variable, see [19].

3. $K(\pi, 1)$–PROBLEM FOR THE COMPLEMENT OF THE ESSENTIAL RAMIFICATION SET OF THE GENERAL REAL POLYNOMIAL IN ONE VARIABLE

Consider the space $\mathbb{R}^d$ of all real polynomials

$$\text{(1)} \qquad f_a(x) \equiv x^d + a_1 x^{d-1} + \cdots + a_{d-1} x + a_d, \quad a_j \in \mathbb{R}.$$

The *essential ramification set* in the space $\mathbb{R}^d$ is the union of all values $a = (a_1, \ldots, a_d)$ such that the corresponding polynomial f_a has either a real triple root, or a pair of complex conjugate imaginary double roots, see [17]. Obviously, this set is a subvariety of codimension 2 in $\mathbb{R}^d$.

Problem 3. *Is its complement a $K(\pi, 1)$-space?*

4. ODD-DIMENSIONAL NEWTON'S LEMMA ON INTEGRABLE OVALS AND GEOMETRY OF HYPERSURFACES

This is actually the "odd-dimensional part" of the Arnold's problem 1987-14 from [2] (repeated as problem 1990-27). I describe below some its reduction to a problem in algebraic geometry.

Any compact domain in $\mathbb{R}^n$ defines a two-valued function on the space of affine hyperplanes: the volumes of two parts into which the hyperplane cuts the domain. If n is odd and the domain is bounded by an ellipsoid, then this function is algebraic (by a generalization of the Archimedes' theorem on sphere sections).

Arnold's problem (see [2]). *Do there exist smooth hypersurfaces in $\mathbb{R}^n$ (other than the quadrics in odd-dimensional spaces), for which the volume of the segment cut by any hyperplane from the body bounded by them is an algebraic function of the hyperplane?*

Many obstructions to the algebraicity of the volume function follow from the *Picard–Lefschetz theory* studying the ramification of integral functions, see [15], [3]. These obstructions are quite different in the case of even or odd n because the homology intersection forms, which are the major part of the Picard–Lefschetz formulas, behave very differently depending on the parity of n. In particular, the "even-dimensional" obstructions are enough to prove that the volume function of a compact domain with C^∞-smooth boundary in $\mathbb{R}^{2k}$ never is algebraic, see [16]. Here are two similar obstructions specific for the case of odd n.

Definition. A non-singular point of a complex algebraic hypersurface is called *parabolic* if the second fundamental form of the hypersurface (or, equivalently, the Hessian matrix of its equation) is degenerate at this point. A parabolic point x is *degenerate* if the tangent hyperplane to out hypersurface at x is tangent to it at entire variety of positive dimension, containing our point.

Proposition (see [15]). *If n is odd and the volume function defined by a bounded domain with smooth boundary in $\mathbb{R}^n$ is algebraic, then the complexification of this boundary cannot have non-degenerate parabolic points in $\mathbb{C}^n$.*

Smooth algebraic projective hypersurfaces of degree ≥ 3 always have parabolic points (and moreover, by a theorem of F. Zak they have only non-degenerate parabolic points). Unfortunately, this is not sufficient to give the negative answer to the above Arnold's problem, because

a) the complexification of a smooth real hypersurface can have singular points in the complex domain, and non-smooth hypersurfaces of arbitrarily high degrees can have no parabolic points: for instance this is the case for hypersurfaces projective dual to smooth ones;

b) the previous proposition does not prohibit parabolic points in the non-proper plane $\mathbb{CP}^n \setminus \mathbb{C}^n$.

However, the standard singular points which can occur instead of parabolic points, the *generic cuspidal edges*, also prevent the algebraicity of the corresponding volume function, see [15], §III.6.

Problem 4. *Are these geometric obstructions sufficient to solve the above problem?*

(That is, is it correct that the complexification of the smooth algebraic boundary of degree ≥ 3 of a compact domain in $\mathbb{R}^n$ always has a point of one of these two obstructing types?) If not, probably we can complete this list by some other singularity types, also obstructing the algebraicity, in such a way that singular points of at least one of these types will be unavoidable on any such hypersurface?

5. Greedy simplifications of real algebraic manifolds

Given natural numbers d and N, consider the space $P(d; N)$ of all smooth algebraic hypersurfaces of degree d in $\mathbb{R}^N$. The *trivial* elements of this space are the empty manifolds if d is even, and the surfaces isotopic to the unknotted $\mathbb{R}^{N-1}$ if d is odd. Consider also some natural measure of topological complexity of such hypersurfaces, such as the sum of generators of homology groups, or the lowest number of critical

points of Morse functions, taking the absolutely minimal value on the trivial objects only.

Problem 5A. *Is it correct that any hypersurface from the space $P(d; N)$ can be connected with a trivial one by a generic path in this space so that it experiences only Morse surgeries, any of which decreases this complexity measure?*

In other words, do there exist non-trivial varieties from our space, any surgery of which increase (or do not change) this complexity measure?

This problem can be extended to algebraic submanifolds defined by systems of polynomials; however the measure of topological complexity in this case should be chosen carefully, taking in account the possible "knottedness" in $\mathbb{R}^N$.

Problem 5B. *A version of the previous problem, when the complexity measure is not purely topological: namely, it is the lowest number of critical points of Morse functions, defined by restrictions of linear functions $\mathbb{R}^N \to \mathbb{R}$ to our varieties. (Correspondingly, the surgeries of the variety affecting this measure are not only of topological nature, but also include bifurcations of the dual variety).*

If the answer to the previous questions is negative, we obtain the functions associating with any value T of topological complexity the lowest number F such that any surface of complexity T can be connected with a trivial one by such a generic path in the space $P(d; N)$ that the complexities of all intermediate hypersurfaces do not exceed F.

Problem 5C. *Give an upper bound for the function $T \mapsto F$.*

6. A LOCAL VERSION OF THE PROBLEM 5

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a function germ with $df(0) = 0$ and finite Milnor number $\mu(f)$. Let $\rho(f)$ be the smallest number of real critical points of real Morsifications of f.

Problem 6A. *Is it correct that any real Morsification of f can be connected with one of complexity $\rho(f)$ by a generic path in the base of a versal deformation in such a way that all Morse surgeries $[A_2]$ in this path only decrease the number of real critical points?*

Problem 6B. *What can be said about the number $\rho(f)$?*

Two obvious lower estimates of it are a) the index of $\mathrm{grad}\, f$ at 0, and b) the Smale number of the relative homology group

(2) $$H_*(f^{-1}((-\infty, \varepsilon]), f^{-1}((-\infty, -\varepsilon]))$$

(i.e. the rank of the free part of this group plus twice the minimal number of generators of its torsion). Of course, the first number does not exceed the second, but *can they be different? Do they coincide at least for functions of corank 2?*

Can the group (2) have a non-trivial torsion? Is the estimate b) of the number $\rho(f)$ sharp?

Problem C *Is it true that any component of the complement of the discriminant variety of a versal deformation contains a Morsification, whose all $\mu(f)$ critical points are real?*

This is true for all simple singularities: see [15].

7. CONVERGENCE RADIUS OF THE MULTIDIMENSIONAL NEWTON'S METHOD

Consider a polynomial $\mathbb{C}^1 \to \mathbb{C}^1$ of degree n and some its simple root z_0. Let d be the minimal distance from this root to all other roots of this polynomial. According to [10], the $\frac{d}{2n-1}$-neighborhood of z_0 belongs to its convergence domain, that is, the Newton's method starting from any point of this neighborhood converges to z_0. This estimate cannot be improved as an universal function in d and n.

Problem 7. *Give similar universal estimate of the radius of convergence domains of the multidimensional Newton's method of* [12].

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